THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
MATH4230 2024-25
Lecture 21
April 2, 2025 (Wednesday)

Chapter 3: Numerical Method

The Projected gradient method

For the problem $\inf_{x \in K} f(x)$, we introduce the following algorithm:

$$\textbf{Algorithm}: \qquad x_{k+1} = \Pi_K \left(x_k - \rho \nabla f(x_k) \right)$$

Definition 1. Let K be a closed set in $\mathbb{R}^n$. We call the **projection** of a point $y \notin K$ the point x^* such that

$$\|x^* - y\| = \min_{x \in K} \|x - y\|.$$

We denote $x^* := \Pi_k(y)$.

Proposition 1. *Let K be a closed convex set in $\mathbb{R}^n$, and $y \notin K$. Then*

- $\Pi_K(y)$ *exists and is uniquely determined.*
- $\Pi_K(y) = x^*$ *is the unique point in K satisfying:*

$$\langle y - x^*, z - x^* \rangle \le 0, \quad \forall z \in K$$

Proof. 1. For the existence, since the map $x \mapsto \|x - y\|^2$ is strictly convex and **coercive** since minimizing a coercive function in a closed set, there exists a minimizer.
Furthermore for the uniqueness, by the strictly convexity, the minimizer x^* is unique.

2. We next prove that "$x^* = \Pi_K(y)$ satisfies $\langle y - x^*, z - x^* \rangle \le 0, \quad \forall z \in K$".
For any $z \in K$ and $t \in (0, 1]$, then $tz + (1-t)x^* \in K$ by the convexity of K.
Then, we have $\|tz + (1-t)x^* - y\|^2 \ge \|x^* - y\|^2$ since x^* is the minimizer.
Now, we consider

$$\begin{aligned} \|tz + (1-t)x^* - y\|^2 &= \|t(z - x^*) + x^* - y\|^2 \\ &= \|x^* - y\|^2 + 2 \langle x^* - y, z - x^* \rangle t + \|z - x^*\|^2 t^2 \end{aligned}$$

So, it follows that

$$\begin{aligned} 2 \langle x^* - y, z - x^* \rangle t + \|z - x^*\|^2 t^2 \ge 0 \\ \langle y - x^*, z - x^* \rangle \le \frac{t}{2} \|z - x^*\|, \ \forall t \in (0, 1] \end{aligned}$$

Since $t > 0$ is arbitrary, so this gives $\langle y - x^*, z - x^* \rangle \le 0$ for all $z \in K$.

3. Let $\hat{x}$ satisfy $\langle y-\hat{x}, z-\hat{x}\rangle \leq 0,\ \forall z \in K$. Now, we aim to prove that $\hat{x} = \hat{x^*}$.
   For all $z \in K$, note that

$$\begin{aligned} \|y-z\|^2 - \|y-\hat{x}\|^2 &= \|y-\hat{x}+\hat{x}-z\|^2 - \|y-\hat{x}\|^2 \\ &= \|y-\hat{x}\|^2 + 2\underbrace{\langle y-\hat{x}, \hat{x}-z\rangle}_{\geq 0} + \|\hat{x}-z\|^2 - \|y-\hat{x}\|^2 \\ &\geq \|\hat{x}-z\|^2 \geq 0 \end{aligned}$$

   This proves that $\|y-z\|^2 \geq \|y-\hat{x}\|^2$ for all $z \in K$, by the uniqueness, this proves that $\hat{x}$ is the minimizer, that is $\hat{x} = \Pi_K(y)$.

□

Theorem 2. *Assume that $f : \mathbb{R}^n \to \mathbb{R}$ is **elliptic** in the sense that*

- $\mathrm{Hess}(f)(x) \geq \alpha I_n$ *for some $\alpha > 0$ and*
- $\nabla f : \mathbb{R}^n \to \mathbb{R}^n$ *is M-Lipschtiz and $\rho \in \left(0, \dfrac{2\alpha}{M^2}\right)$. Then $x_k \to x^*$, where x^* is the minimizer of the problem $\min\limits_{x\in K} f(x)$.*

Remarks. We define $\mathrm{Hess}(f) = \left(\partial^2_{x_i x_j} f(x)\right)_{i,j=1,\ldots,n}$.
We say $\mathrm{Hess}(f) \geq \alpha I_n$ if and only if $\mathrm{Hess}(f) - \alpha I_n$ is semi positive definite.

Example 1. When $n = 1$, $f(x) = a_2 x^2 + a_1 x + a_0$ with $a_2 > 0$, then it satisfies the condition.

Lemma 3. *Under the same condition, let x^* be the minimizer, then we have*

$$\langle \nabla f(y) - \nabla f(x^*), y - x^*\rangle \geq \alpha \|y - x^*\|^2$$

and $\Pi_K(x^* - \rho\nabla f(x^*)) = x^*$.

Remarks. We will complete the proof for this lemma in next week.

— End of Lecture 21 —